Solution to Assignment 3

29. Find the area of one leaf of the rose $r = 12\cos 3\theta$.

Solution. As the cosine function is 2π-periodic, $\cos 3\theta$ is $2\pi/3$-periodic. It suffices to plot its graph in $[-\pi/3, \pi/3]$. Observing that in this interval, $\cos 3\theta$ is non-negative only on $[-\pi/6, \pi/6]$, there is one leaf sitting in $[-\pi/6, \pi/6]$. By rotating it by $2\pi/3$ and then by $4\pi/3$, we obtain the full graph of the rose which consists of three identical leaves.

By symmetry, the area of one leaf is

$$\int_{-\pi/6}^{\pi/6}\int_0^{12\cos 3\theta} r\,drd\theta = 2\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,drd\theta = 12\pi\ .$$

41. In this problem we establish the famous formula by using double integral in a tricky way. Setting

$$a = \int_{-\infty}^{\infty} e^{-x^2}\,dx\ .$$

We have

$$\begin{aligned}
a^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\
&= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA(x,y) \\
&= \lim_{R\to\infty}\iint_{D_R} e^{-x^2-y^2}\,dA(x,y) \\
&= \lim_{R\to\infty}\int_0^{2\pi}\int_0^{R} e^{-r^2} r\,drd\theta \\
&= \lim_{R\to\infty}\int_0^{2\pi}\int_0^{R^2} e^{-s}\,dsd\theta \\
&= \pi.
\end{aligned}$$

Hence

$$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\ .$$

Supplementary Problems

1. Let D be the region bounded by $y = x^2$ and $y = 2$. Express D in polar coordinates. Hint: Decompose D into three regions.

   Solution. The curves $y = x^2$ and $y = 2$ intersect at the origin and $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$. The region D is the union of the following three regions.

   $$D_1 : 0 \le \theta \le \theta_0, \quad 0 \le r \le \tan\theta\sec\theta,$$

   $$D_2 : \theta_0 \le \theta \le \pi - \theta_0, \quad 0 \le r \le 2/\sin\theta\ ,$$

   and

   $$D_3 : \pi - \theta_0 \le \theta \le \pi, \quad 0 \le r \le \tan\theta\sec\theta\ .$$

Note that $y = x^2$ is $r = \tan\theta \sec\theta$ and $y = 2$ is $r = 2/\sin\theta$ in polar coordinates. Also, $\theta_0 = \arctan 2/\sqrt{2} = \arctan\sqrt{2}$ is the angle between the line from the origin to $(\sqrt{2}, 2)$ and the positive axis.

2. Let D be the region described by $0 \le x \le 1, 0 \le y \le \sqrt{1-x^2} + 1$. Describe it in polar coordinates. Hint: Decompose D into two regions.

   Solution. As a polar curve, $x^2 + (y-1)^2 = 1$ is given by $r = 2\sin\theta$. This graph and the line $x = 1$ intersect at $(1, 1)$. D is the union of D_1 and D_2 where

   $$D_1 : \pi/4 \le \theta \le \pi/2,\ 0 \le r \le 2\sin\theta\ ,$$

   and

   $$D_2 :\ 0 \le \theta \le \pi/4,\ 2\sin\theta \le r \le 1/\cos\theta\ .$$